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Singular Lagrangians affine in velocities

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Abstract
The properties of Lagrangians affine in velocities are analysed in a geometric way. These systems are necessarily singular and exhibit, in general, gauge invariance. The analysis of constraint functions and gauge symmetry leads us to a complete classification of such Lagrangians.

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1. Introduction

The Dirac method of dealing with constrained systems [1–3], developed when looking for a way of quantizing such systems by means of a 'canonical quantization'-like procedure, is shown to be a very efficient method and the geometric approach to the theory of systems defined by singular Lagrangians deserved a large amount of attention during the past years.

In particular, Lagrangians that are affine in time derivatives have been analysed [4] in the framework of pre-symplectic geometry and the geometric theory of time-independent singular Lagrangians [5]. They are important because their Euler–Lagrange equations become systems of first-order differential equations, appearing as constraints, instead of systems of second order as it would happen with regular Lagrangians. So they will play a relevant role in many cases, not only in physics, where many equations are first order, as in the Dirac equation, but also in other fields as in biology dynamics and chemistry. On the other hand, these systems are singular and then they give rise to gauge ambiguities and gauge symmetries.

The geometric study of a particular type of singular Lagrangians, those which are affine in velocities, was carried out in [4] in the framework of autonomous systems, with the aim of studying the inverse problem of Lagrangian mechanics and the theory of non-point symmetries from a new geometric perspective. Almost simultaneously, Faddeev and Jackiw developed a method for the quantization of such singular systems which soon became very popular and received much attention from most theoretical physicists. This procedure of dealing with such systems is usually referred to as Faddeev–Jackiw (FJ) method of quantization [6–12].

One of the most important properties of singular Lagrangians is the existence of infinitesimal gauge symmetries which are related to the second Noether theorem. This, which is particularly important for field theories, has sense only in the framework of time-dependent systems, with time playing the role of base coordinates in field theory. A geometric approach to second Noether theorem was given in [13], but the geometric theory of the time-dependent description of such affine velocities in Lagrangian systems has never been developed, as far as we know, even if it is very important as the only way of fully understanding the meaning of Noether's second theorem establishing the relationship between singular Lagrangians and gauge transformations. Then we feel that a re-examination of the problem of singular Lagrangians that are first order in velocities will be very useful and will allow a better understanding of the theory.

The two fundamental aspects of these systems described by these first order in velocities Lagrangians are the presence of first-order equations and constraints, which make possible this alternative Faddeev–Jackiw method of quantization [6, 14]. This method is based on the reduction theory for the pre-symplectic form defined by the singular Lagrangian, but it admits an alternative by means of the addition of the constraints with some Lagrange multipliers in order to obtain the symplectic extension [11, 12, 15, 16]. Once the symplectic structure is obtained and, therefore, Poisson brackets are defined, we can make use of the canonical quantization procedure. In particular, the FJ-method uses the reduction technique and has been applied in many different fields, from condensed matter [17–19] and astrophysics [20], to fluid dynamics [21] and, of course, high-energy physics [22]. Even the Schrödinger equation itself can be derived using this method [23].

It is also to be noted that the FJ-method can be generalized to also include fermionic degrees of freedom, i.e. non-commutative variables [24] and the corresponding canonical quantization can also be used in super-symmetric theories [8, 25, 26] with applications in super-gravity (see e.g. [27]).

The paper is organized as follows. In section 2 we summarize the results of [4]. A Lagrangian approach to Hamilton equations from a geometric perspective will be given in section 3. The framework for the geometric description of time-dependent singular systems as given in [28] will be indicated in section 4, where we will also include a recipe obtained from [13] (see also [28]) for the search of gauge symmetries for singular Lagrangian systems. The geometric theory of time-dependent Lagrangians which are affine in the velocities will be developed in section 5 and the gauge invariance of such systems will be studied in section 6. Finally, the theory is illustrated with several examples.

As a matter of notation, tangent and cotangent bundles will be denoted by $\tau_M : \mathrm{T}M \to M$ and $\pi_M : \mathrm{T}^*M \to M$, respectively. The set of vector fields along a map $f : M \to N$ (see e.g. [29]), also called f^*-derivations in [30], i.e. maps $X : M \to \mathrm{T}N$ such that $\tau_N \circ X = f$, will be denoted by $\mathfrak{X}(f)$. Examples of such kind of vector fields are $\mathrm{T}f \circ Y$ and $Z \circ f$, where $Y \in \mathfrak{X}(M), \mathrm{T}f : \mathrm{T}M \to \mathrm{T}N$ is the tangent map corresponding to f, and $Z \in \mathfrak{X}(N)$. It has been shown by Pidello and Tulczyjew [30] that a vector field X along f determines two f^*-derivations of scalar differential forms on M—one of type i_* and degree -1, denoted by i_X, and the other of type d_*, denoted by d_X, defined in the following way: given $\alpha \in \bigwedge^p(N)$ and $v_1, \ldots, v_{p-1} \in \mathrm{T}_xM$, we define $i_X\alpha \in \bigwedge^{p-1}(M)$ by $\langle i_X\alpha(x), (v_1, \ldots, v_{p-1})\rangle = \langle \alpha(f(x)), (X(m), f_{*x}(v_1), \ldots, f_{*x}(v_{p-1}))\rangle$, and $d_X\alpha \in \bigwedge^p(M)$ is given by $d_X\alpha = \mathrm{d}i_X\alpha + i_X\,\mathrm{d}\alpha$ [31].

2. Geometric approach to time-independent Lagrangians which are affine in velocities

In the geometric description of a time-independent Lagrangian system, the states are described by points of the tangent bundle $\tau_Q : \mathrm{T}Q \to Q$ of the configuration space Q, which is assumed to be an n-dimensional differentiable manifold. We are interested in the geometric study of systems described by a Lagrangian including only terms up to first order in velocities, namely, with a coordinate expression

$$L(q, v) = m_j(q)v^j - V(q) \tag{2.1}$$

where summation on repeated indices is understood.

The property of a function being linear in the fibre coordinates (velocities) of the tangent bundle is intrinsic because it is preserved under point transformations. If $\phi \in C^\infty(\mathrm{T}Q)$ takes the form $\phi = \phi_i(q)v^i$ in a particular set of local coordinates, then, under the change of coordinates $q'^i = q'^i(q)$, we have that in order to have

$$\phi_i(q)v^i = \phi_i'(q')\left(\frac{\partial q'^i}{\partial q^j}\right)v^j$$

the components $\phi_i(q)$ should transform as

$$\phi_j(q) = \phi_i'(q')\left(\frac{\partial q'^i}{\partial q^j}\right) \qquad j = 1, \dots, n \tag{2.2}$$

i.e., the functions $\phi_i(q)$ should transform as the components of an associated basic 1-form $\alpha \in \bigwedge^1(\mathrm{T}Q)$, $\alpha = \phi_i(q)\,\mathrm{d}q^i$. This suggests that there is thus a one-to-one linear correspondence between basic 1-forms and linear homogeneous functions that we establish next: if $\mu \in \bigwedge^1(Q)$ is a 1-form, then $\hat{\mu}$ denotes the function $\hat{\mu} \in C^\infty(\mathrm{T}Q)$ defined by $\hat{\mu} = i_{\mathbf{t}}\mu$, where $\mathbf{t} \in \mathfrak{X}(\tau_Q)$ is the identity map id: $\mathrm{T}Q \to \mathrm{T}Q$ viewed as a vector field along the tangent bundle projection τ_Q; in fibred coordinates $\mathbf{t} = v^i\partial/\partial q^i$. That is, $\hat{\mu}(q, v) = \langle \mu(q), \mathbf{t}(q, v)\rangle$, the contraction making sense because $\mathbf{t}(q, v) \in \mathrm{T}_qQ$. In coordinates, when $\mu = m_j(q)\,\mathrm{d}q^j$, $\hat{\mu}(q, v) = m_i(q)v^i$.

The geometric theory for systems described by a Lagrangian $L = \hat{\mu} - \tau_Q^*V$, with $V \in C^\infty(Q)$, that in coordinates of the bundle induced from those of a chart in Q becomes $L = m_j(q)v^j - V(q)$, has been studied in [4]: the energy E_L and the pre-symplectic form ω_L are given by $E_L = \tau_Q^*(V)$, $\theta_L = \tau_Q^*\mu$, and, therefore, $\omega_L = -\tau_Q^*(\mathrm{d}\mu)$, which in coordinates reads as follows:

$$\omega_L = \left(\frac{\partial m_i}{\partial q^k} - \frac{\partial m_k}{\partial q^i}\right)\mathrm{d}q^i \wedge \mathrm{d}q^k.$$

The Hessian matrix W with elements $W_{ij} = \partial^2 L/\partial v^i \partial v^j$ vanishes identically and, therefore, all the τ_Q-vertical vectors, i.e. $\xi^i(q, v)\partial/\partial v^i$, are in the kernel of ω_L.

The search for the other elements in the kernel of ω_L starts by looking for a local basis $\{Z_a = (z_a^i)\}$ of the module of eigenvectors (if any) corresponding to the null eigenvalue of the matrix A with elements given by elements

$$A_{ij} = \frac{\partial m_j}{\partial q^i} - \frac{\partial m_i}{\partial q^j} \qquad i, j = 1, \dots, n. \tag{2.3}$$

Then, a basis for the kernel of ω_L is generated by the vector fields $X_a = z_a^i(q)\partial/\partial q^i$ and $\partial/\partial v^i$, with $a = 1, \dots, n - n_0$, where n_0 denotes the rank of the matrix A. The primary constraint submanifold $\mathcal{M}$ is then determined by the constraint functions (see e.g. [5]) $\phi_X = XE_L$, with $X \in \operatorname{Ker}\omega_L$, which in the present case are $\phi_a = X_aE_L$, i.e.

$$\phi_a = z_a^i(q)\frac{\partial V}{\partial q^i} \tag{2.4}$$

because the energy is τ_Q-projectable, $E_L = \tau_Q^*V$, and then $\partial E_L/\partial v^i \equiv 0$.

The general solution for the dynamical equation $i(X)\omega_L = \mathrm{d}E_L$ will be given by

$$X = \left[\eta^i + \lambda^a z_a^i\right]\frac{\partial}{\partial q^i} + f^i\frac{\partial}{\partial v^i} \tag{2.5}$$

with $\eta^i(q)$ being a solution of

$$A_{ij}\eta^j = \frac{\partial V}{\partial q^i} \tag{2.6}$$

and where λ^a and f^i are arbitrary functions on TQ.

There is a special class of vector fields D in TQ which are called second-order vector fields, hereafter shortened as SODE fields, which are characterized by $S(D) = \Delta$, where S denotes the vertical endomorphism [32–34] and Δ is the Liouville vector field generating dilations along the fibres. They can also be characterized by $\mathrm{T}\,\tau_Q \circ D = \mathbf{t}$.

The constraint functions for a Lagrangian given by (2.1) are basic functions, i.e. holonomic constraints, defining a submanifold Q' of Q. Consequently, the secondary constraint functions for the existence of a solution restriction of second-order vector field will be $\widehat{\mathrm{d}\phi_a}$ and are given by linear functions in the velocities. A solution X given by (2.5) can be the restriction of a SODE only in those points of TQ for which

$$A_{ij}v^j = \frac{\partial V}{\partial q^i}. \tag{2.7}$$

In these points, the general solution of the dynamical equation is given by

$$X = \left[v^i + \lambda^a z_a^i\right]\frac{\partial}{\partial q^i} + f^i\frac{\partial}{\partial v^i} \tag{2.8}$$

while the SODE condition corresponds to the choice $\lambda^a = 0$.

The particularly simple case in which $(Q,\, \mathrm{d}\mu)$ is a symplectic manifold, i.e. $\det A \neq 0$, and therefore Q is even dimensional, was also studied in [4], where it was shown that then $\operatorname{Ker}\omega_L$ is made up of all τ_Q-vertical vectors and therefore there will be no dynamical constraint function. The globally defined solution of the dynamical equation is

$$X = \eta^i\frac{\partial}{\partial q^i} + f^i\frac{\partial}{\partial v^i}$$

with the functions η^i uniquely determined by

$$\eta^i = (A^{-1})^{ij}\frac{\partial V}{\partial q^j}. \tag{2.9}$$

The Marsden–Weinstein theory of reduction for the pre-symplectic system defined by the Lagrangian (2.1), $(\mathrm{T}Q, \omega_L = -\tau_Q^*(\mathrm{d}\mu), E_L = \tau_Q^*(V))$, establishes that the reduced symplectic manifold is $(Q, -\mathrm{d}\mu)$ with Hamiltonian function V.

The Hamiltonian formalism for the Lagrangian (2.1) was also studied in [4]: the primary constraint submanifold $P_1 = \mathcal{F}L(\mathrm{T}Q)$ is determined by $\Phi_j(q,p) = p_j - m_j(q) = 0$, i.e. the graph of the form μ. Using the identification of it with the base Q the pull-back of the canonical 1-form ω_0 in T^*Q is $-\mathrm{d}\mu$. The Poisson brackets of the constraint functions are $\{\Phi_j, \Phi_k\} = \{p_j - m_j(q), p_k - m_k(q)\} = A_{jk}$, and therefore, when $\mathrm{d}\mu$ is symplectic, all the constraints are of the second class. The Hamiltonian function H is defined on P_1 by the restriction of the function $\tilde{V} = \pi_Q^* V$. The general theory leads again to the study of the Hamiltonian dynamical system $(Q, -\mathrm{d}\mu, V)$, as in the Lagrangian case.

Let now L be a singular Lagrangian for which all constraint functions $\phi_a = X_a E_L$, $a = 1, \ldots, n - n_0 = N$, are holonomic; we consider an extended configuration space $\mathbb{R}^N \times Q$ and denote by pr_2 the natural projection $\mathrm{pr}_2 : \mathbb{R}^N \times Q \to Q$. We can then introduce a new Lagrangian $\mathbf{L}_1$ in $\mathrm{T}(\mathbb{R}^N \times Q)$ by

$$\mathbf{L}_1 = \tilde{L} + \lambda^a\tilde{\phi}_a \tag{2.10}$$

where the tilde stands for the $\mathrm{T}\,\mathrm{pr}_2$-pull-back and λ^a are the new additional coordinates (whose corresponding velocities will be represented by ζ^a). Taking into account that

$$\omega_{\mathbf{L}_1} = \tilde{\omega}_L \qquad E_{\mathbf{L}_1} = \tilde{E}_L - \lambda^a \tilde{\phi}_a \tag{2.11}$$

we see that $\operatorname{Ker}\omega_{\mathbf{L}_1}$ is generated by the set of vector fields projecting onto the vector fields X_a of $\operatorname{Ker}\omega_L$, plus $\partial/\partial\lambda^a$ and $\partial/\partial\zeta^a$. The constraint functions determined by $\partial/\partial\lambda^a$ are the (pull-back of the) original primary constraint functions ϕ_a and SODE condition leads us to consider the tangent bundle of the new configuration space $\mathbb{R}^k \times Q'$. The solutions of the dynamics will project under pr_2 onto the solutions of the original problem.

A similar approach can be adopted when we use $\widehat{\mathrm{d}\phi_a}$ instead of ϕ_a as constraint functions and we replace the original Lagrangian L for $\mathbf{L}_2 = \tilde{L} + \lambda^a \widetilde{\widehat{\mathrm{d}\phi_a}}$. In this case $E_{\mathbf{L}_2} = \tilde{E}_L$ and $\omega_{\mathbf{L}_2} = \tilde{\omega}_L + \mathrm{d}\tilde{\phi}_a \wedge \mathrm{d}\lambda^a$, from which we can see that

$$X = \xi^i \frac{\partial}{\partial q^i} + \xi^a \frac{\partial}{\partial \lambda^a} + \eta^i \frac{\partial}{\partial v^i} + \eta^a \frac{\partial}{\partial \zeta^a}$$

is in $\operatorname{Ker}\omega_{\mathbf{L}_2}$ if and only if

$$A\xi - \xi^a \nabla\phi_a - W\eta = 0 \qquad \xi \cdot \nabla\phi^a = 0 \qquad W\xi = 0$$

from which we see that we will obtain as constraint functions (the pull-back of) those obtained directly from L, and the dynamics will correspond, up to the gauge ambiguity in the coordinates λ^a, to the dynamics obtained in Q'.

The relation $\lambda^a \widehat{\mathrm{d}\phi_a} = \widehat{\mathrm{d}(\lambda^a \phi_a)} - \zeta^a \phi_a$ shows that the Lagrangian $\mathbf{L}_2$ may be replaced by $\mathbf{L}_3 = \tilde{L} - \zeta^a \tilde{\phi}_a$, which is quite similar to $\mathbf{L}_1$ with the change of λ^a for its velocity ζ^a.

In the more general situation for which $\mathrm{d}\mu$ is singular, the primary constraint functions (2.4) will be holonomic and therefore the previous remarks show us that they will determine a submanifold $Q' \subset Q$ characterized by some constraint functions ϕ_a and the corresponding secondary constraint functions for a second-order evolution will be $\widehat{\mathrm{d}\phi_a}$. Every such constraint can be incorporated in a new Lagrangian $\mathbf{L}$ defined in the tangent bundle of a new configuration space $\mathbf{Q}$ of the form $\mathbf{Q} = Q \times \mathbb{R}^{(n-n_0)}$, by

$$\mathbf{L}(q^i, \lambda, v^i, \zeta) = L(q^i, v^i) + \lambda^a\, \widehat{\mathrm{d}\phi}_a(q^i, v^i) \tag{2.12}$$

where $(q^i, \lambda^a, v^i, \zeta^a)$ denote the coordinates on the tangent bundle $\mathrm{T}\mathbf{Q}$.

The expressions for $\theta_{\mathbf{L}}$ and $\omega_{\mathbf{L}}$ are

$$\theta_{\mathbf{L}} = \tilde{\theta}_L + \lambda^a\, \widetilde{\mathrm{d}\phi_a} \Longrightarrow \omega_{\mathbf{L}} = \tilde{\omega}_L + \widetilde{\mathrm{d}\phi_a} \wedge \mathrm{d}\lambda^a \tag{2.13}$$

and therefore the rank of $\omega_{\mathbf{L}}$ may be higher than that of ω_L and this is the starting point in the Faddeev–Jackiw approach. The energy $E_{\mathbf{L}}$ is the pull-back of E_L, $E_{\mathbf{L}} = \tilde{E}_L$.

3. A Lagrangian approach to Hamilton equations

Let M be the configuration space of a mechanical system and consider $Q = \mathrm{T}^*M$ endowed with its exact canonical symplectic structure $\omega = -\mathrm{d}\theta_0$, where θ_0 is the Liouville 1-form in T^*M (see e.g. [35]). Then, given a function $H \in C^\infty(\mathrm{T}^*M)$, let us define the linear Lagrangian $L \in C^\infty(\mathrm{T}(\mathrm{T}^*M))$ by

$$L = \hat{\theta}_0 - \tau^*_{\mathrm{T}^*M} H \tag{3.1}$$

which in local coordinates is written as

$$L(q, p; v, u) = p_j v^j - H(q, p). \tag{3.2}$$

In this case $\theta_L \bigwedge^1(\mathrm{T}(\mathrm{T}^*M))$ is given by $\theta_L = \tau^*_{\mathrm{T}^*M}\theta_0$ and $E_L = \tau^*_{\mathrm{T}^*M}H$. The matrix A given by (2.3) is now the symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The kernel of $\omega_L = -\mathrm{d}\theta_L$ is then made up of the τ_{T^*M}-vertical vector fields

$$Y_{f,g} = f^i(q,p;v,u)\frac{\partial}{\partial v^i} + g^i(q,p;v,u)\frac{\partial}{\partial u^i} \tag{3.3}$$

and therefore the pre-symplectic system defined by L, $(\mathrm{T}(\mathrm{T}^*M), \omega_L, E_L)$, admits a global dynamics which is not uniquely defined but given by

$$\Gamma_{f,g} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + f^i(q,p;v,u)\frac{\partial}{\partial v^i} + g^i(q,p;v,u)\frac{\partial}{\partial u^i} \tag{3.4}$$

where f and g are arbitrary functions. The integral curves of each one of such vector fields are determined by the system of differential equations,

$$\begin{cases} \dot q^i = \frac{\partial H}{\partial p_i} \\ \dot p_i = -\frac{\partial H}{\partial q^i} \\ \dot u^i = f^i(q,p;v,u) \\ \dot v^i = g^i(q,p;v,u). \end{cases}$$

However, only in those points for which

$$\begin{cases} v^i = \frac{\partial H}{\partial p_i} \\ u^i = -\frac{\partial H}{\partial q^i} \end{cases} \tag{3.5}$$

the solution can be chosen to be the restriction of a SODE vector field. The preceding equations determine a submanifold C on $\mathrm{T}(\mathrm{T}^*M)$ and the condition on the restriction of the vector field $\Gamma_{f,g}$ to be tangent to C determines the functions f^i and g^i by means of

$$f^i = \Gamma\left(\frac{\partial H}{\partial p_i}\right) \qquad g^i = -\Gamma\left(\frac{\partial H}{\partial q^i}\right) \tag{3.6}$$

obtaining in this way the vector field on C,

$$\Gamma_C = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \Gamma\left(\frac{\partial H}{\partial p_i}\right)\frac{\partial}{\partial v^i} - \Gamma\left(\frac{\partial H}{\partial q^i}\right)\frac{\partial}{\partial u^i}. \tag{3.7}$$

The dimension of C is only twice that of M and then it can be parametrized by (q^i, p_i). The expression of Γ_C in these new coordinates is

$$\Gamma_C = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

and therefore, here the Hamilton equations arise as determining the integral curves of the uniquely defined vector field in the submanifold C in which such a SODE solution of the dynamical equation can be found. Then, a curve $\gamma : I \to M$ whose lift to $\mathrm{T}M$ lies in C is the solution we were looking for.

4. Geometric description of time-dependent singular system

For the reader's convenience, we introduce in this section the notation to be used and a summary of several properties and results of interest for the following sections.

The evolution space of a time-dependent mechanical system whose configuration space is the n-dimensional manifold Q is $\mathbb{R} \times \mathrm{T}Q$ [36], which is the space of 1-jets of the trivial bundle $\pi : \mathbb{R} \times Q \longrightarrow \mathbb{R}$, $J^1\pi = \mathbb{R} \times \mathrm{T}Q$; $\mathbb{R}$ is endowed with a volume form $\mathrm{d}t \in \bigwedge^1(\mathbb{R})$ and represents the Newtonian time. There is one vector field in $\mathbb{R}$, $\mathrm{d}/\mathrm{d}t \in \mathfrak{X}(\mathbb{R})$, such that $i(\mathrm{d}/\mathrm{d}t)\,\mathrm{d}t = 1$. The main geometric tools to be used in the geometric description of time-dependent mechanics are those of jet bundle geometry [37, 38]. The k-jet bundle of π is $J^k\pi = \mathbb{R} \times \mathrm{T}^kQ$, with T^kQ representing the space of k-velocities. In particular, $J^2\pi = \mathbb{R} \times \mathrm{T}^2Q$ is the space of accelerations and $J^1\pi = \mathbb{R} \times \mathrm{T}Q$, the space of velocities. By convention we will write $J^0 \equiv \mathbb{R} \times Q$. For each pair of indices k, l such that $k > l$, there is a natural projection $\pi_{k,l} : \mathbb{R} \times \mathrm{T}^kQ \longrightarrow \mathbb{R} \times \mathrm{T}^lQ$ and we will denote $\pi_k = \pi \circ \pi_{k,0} : \mathbb{R} \times \mathrm{T}^kQ \longrightarrow \mathbb{R}$, the projection of $J^k\pi$ onto $\mathbb{R}$.

If $\sigma \in \mathrm{Sec}(\pi)$, then $j^k\sigma \in \mathrm{Sec}(\pi_k)$ will denote the k-jet prolongation of σ. So, if $\sigma(t) = (t, \sigma^i(t))$, we have $j^k\sigma(t) = (t, \sigma^i(t), \mathrm{d}\sigma^i/\mathrm{d}t, \ldots, \mathrm{d}^k\sigma^i/\mathrm{d}t^k)$. We also recall that the differential 1-forms $\theta \in \bigwedge^1(J^k\pi)$ such that $(j^k\sigma)^*\theta = 0$, no matter the section σ, are called contact 1-forms of $J^k\pi$. They are the constraint 1-forms for the so-called Cartan distribution.

The theory of time-dependent Lagrangian systems makes an extensive use of the notion of vector field along a map. In particular, there exist vector fields along $\pi_{k+1,k}$, $\mathbf{T}^k$, representing the total derivative operators. They are defined by means of $\mathbf{T}^k \circ j^{k+1}\sigma = \mathrm{T}(j^k\sigma) \circ \mathrm{d}/\mathrm{d}t$, $\forall\sigma \in \mathrm{Sec}(\pi)$. For every $F \in C^\infty(\mathbb{R} \times \mathrm{T}^kQ)$, the Lie derivative of F with respect to $\mathbf{T}^k$, i.e. the function $\mathrm{d}_{\mathbf{T}^k}F = i_{\mathbf{T}^k}\,\mathrm{d}F \in C^\infty(\mathbb{R} \times \mathrm{T}^{k+1}Q)$, represents the total time derivative of F, usually written as $\mathrm{d}F/\mathrm{d}t$ or simply $\dot{F}$. In fibred coordinates (t, q^i, v^i) for $\mathbb{R} \times \mathrm{T}Q$ and the corresponding ones, (t, q^i, v^i, a^i), for $\mathbb{R} \times \mathrm{T}^2Q$, we have $\mathbf{T}^0 = \partial/\partial t + v^i\partial/\partial q^i \in \mathfrak{X}(\pi_{1,0})$ and $\mathbf{T}^1 = \partial/\partial t + v^i\partial/\partial q^i + a^i\partial/\partial v^i \in \mathfrak{X}(\pi_{2,1})$. Obviously $\mathbf{T}^0$ and the operator $\mathbf{t} \in \mathfrak{X}(\tau_Q)$ introduced in section 2 are related, $\mathrm{T}\rho \circ \mathbf{T}^0 = \mathbf{t} \circ \rho_2$, with ρ and ρ_2 being the projections onto the second factor of $\mathbb{R} \times Q$ and $\mathbb{R} \times \mathrm{T}Q$, respectively.

A vector field $X \in \mathfrak{X}(\mathbb{R} \times Q)$ can be lifted to $\mathbb{R} \times \mathrm{T}Q$ giving rise to a unique vector field $X^1 \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}Q)$ which is $\pi_{1,0}$-projectable onto X and preserves the set of contact 1-forms of $\mathbb{R} \times \mathrm{T}Q$. If the coordinate expression of X is $X = \tau\partial/\partial t + X^i\partial/\partial q^i$, then

$$X^1 = \tau\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + (\dot{X}^i - v^i\dot{\tau})\frac{\partial}{\partial v^i}.$$

Vector fields of type X^1 are called infinitesimal contact transformations (hereafter ICT); it is worth noting that

$$(fX)^1 = (\pi_{1,0}^*f)X^1 + \dot{f}X^V$$

where $X^V = S(X^1)$. Here S is the vertical endomorphism on $\mathbb{R} \times \mathrm{T}Q$ [31], which is a (1, 1)-tensor field whose expression in the fibred coordinates (t, q^i, v^i) is $S = \partial/\partial v^i \otimes (\mathrm{d}q^i - v^i\,\mathrm{d}t)$. Note that the local 1-forms given by $\theta^i = \mathrm{d}q^i - v^i\,\mathrm{d}t$ generate the set of contact 1-forms of $\mathbb{R} \times \mathrm{T}Q$ and the Cartan distribution is but $\mathrm{Ker}\,S$.

A similar definition works for the prolongation of vector fields $X = \tau(t, q, v)\partial/\partial t + X^i(t, q, v)\partial/\partial q^i \in \mathfrak{X}(\pi_{1,0})$: the vector field along $\pi_{2,1}$ given by

$$X^1 = \tau(t, q, v)\frac{\partial}{\partial t} + X^i(t, q, v)\frac{\partial}{\partial q^i} + (\dot{X}^i - v^i\dot{\tau})\frac{\partial}{\partial v^i}$$

is the *first prolongation of* X. Thus $\mathbf{T}^1 = (\mathbf{T}^0)^1$.

More details about these notions and constructions can be found in [28].

The key object on which the geometric formulation of the dynamics corresponding to a time-dependent Lagrangian $L \in C^\infty(\mathbb{R}\times \mathrm{T}Q)$ is based is the Poincaré–Cartan 1-form, defined by

$$\Theta_L = \mathrm{d}L \circ S + L\,\mathrm{d}t \in \bigwedge^1(\mathbb{R}\times \mathrm{T}Q). \tag{4.1}$$

In fibred coordinates $\Theta_L = (\partial L/\partial v^i)\theta^i + L\,\mathrm{d}t$. Another important object related with L is the Euler–Lagrange 1-form, which is defined by $\delta L = i_{\mathbf{T}^1}\,\mathrm{d}\Theta_L \in \bigwedge^1(\mathbb{R}\times \mathrm{T}^2Q)$, with local coordinate expression $\delta L = L_i\theta^i$, where

$$L_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \pi^*_{2,1}\left(\frac{\partial L}{\partial q^i}\right)$$

are the variational derivatives of the Lagrangian L.

The dynamical equation to be considered here is

$$i(\Gamma)\,\mathrm{d}\Theta_L = 0 \tag{4.2}$$

and, according to Hamilton's principle, the condition for the section $\sigma \in \mathrm{Sec}(\pi)$ to be an extremal of the action is

$$(j^1\sigma)^*[i(Z)\,\mathrm{d}\Theta_L] = 0 \qquad \forall Z \in \mathfrak{X}(\mathbb{R}\times \mathrm{T}Q).$$

The problem is to find a vector field $\Gamma \in \mathfrak{X}(\mathbb{R}\times \mathrm{T}Q)$ which is solution of the dynamical equation (4.2) and whose integral curves are the first prolongation $j^1\sigma$ of sections $\sigma \in \mathrm{Sec}(\pi)$. In other words, Γ must be a time-dependent SODE, i.e. a vector field Γ such that $\mathrm{T}\pi_{1,0}\circ\Gamma = \mathbf{T}^0$. Its integral curves are parametrized by t and the local expression of Γ is $\Gamma = \partial/\partial t + v^i\partial/\partial q^i + \Gamma^i\partial/\partial v^i$. Both SODE-type and ICT-type fields belong to $\mathrm{Ker}\,S$, but in general a SODE is not an ICT.

The submanifold of $\mathbb{R}\times \mathrm{T}Q$ where the dynamical equation (4.2) possesses such kind of solutions is given by the following theorem [28]:

Theorem 1. *Let M_L denote the coisotropic subbundle*

$$M_L = \{Z \in \mathrm{T}(\mathbb{R}\times \mathrm{T}Q) | S(Z) \in \mathcal{V}_{1,0}(\mathrm{Ker}\,\mathrm{d}\Theta_L)\}$$

and $\mathcal{V}_{1,0}(\cdot)$ means the $\pi_{1,0}$-vertical part, which is but the kernel of the differential of the Legendre transformation $\mathcal{F}L : \mathbb{R}\times \mathrm{T}Q \longrightarrow \mathbb{R}\times \mathrm{T}^*Q$, that is, $\mathcal{V}_{1,0}(\mathrm{Ker}\,\mathrm{d}\Theta_L) = \mathrm{Ker}\,\mathrm{T}\mathcal{F}L$.

Then, there exist solutions of the dynamical equation which are restrictions of a SODE *field in the points, and only in that points, of the set defined by*

$$\mathcal{M} = \{z \in \mathbb{R}\times \mathrm{T}Q \mid \mathrm{d}\Theta_L(D, Z)(z) = 0, \forall Z \in M_L, D \text{ any SODE}\}.$$

When L is regular, $M_L = \{0\}$ and $\mathcal{M} = \mathbb{R}\times \mathrm{T}Q$ and, consequently, there is no restriction on the motion. But there exist (primary) constraints for a singular Lagrangian which are given by the following conditions:

$$\Phi_Z = \mathrm{d}\Theta_L(D, Z) = 0 \qquad Z \in M_L. \tag{4.3}$$

The functions Φ_Z are the *primary constraint functions*. Obvious conditions for the consistency of the dynamics Γ compatible with the constraints (4.3) are given by

$$\chi_Z = \Gamma(\Phi_Z)|_{\mathcal{M}} = 0 \tag{4.4}$$

which either give rise to the secondary constraints or (partially) fix the dynamics Γ. When the process is iterated, we will hopefully arrive at the final constraint submanifold $\mathcal{M}_f$, on which there exist solutions of the dynamical equation which are the restriction onto $\mathcal{M}_f$ of SODE fields tangent to $\mathcal{M}_f$.

Note that here the only ingredient is the singular Lagrangian L which provides both the (nonholonomic) constraint functions and the dynamics. In this sense this is a problem that does not coincide with the more frequently studied constrained situation in which the starting point is a given nonholonomic constraint distribution [39]. This latter situation, that is receiving much attention during the last years (see e.g. [40, 41]) and is important by its relation with the theory of connections and by its control theoretical applications, is different from the present case of first-order singular Lagrangians.

The dynamics obtained by applying the constraint algorithm sketched above may not be unique, a fact which is related with the gauge invariance of the Lagrangian. The appropriate geometric tool to deal with a gauge infinitesimal transformation

$$\delta q^i = \epsilon \sum_{\alpha=0}^{R} \frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha} X^i_\alpha(t, q, v) \qquad \delta v^i = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q^i) \tag{4.5}$$

where $g = g(t)$ is an arbitrary function of the time, is that of a vector field along the bundle map $\pi_{1,0}$. In fact, let $\left\{X_\alpha = X^i_\alpha(t, q, v)\partial/\partial q^i \mid \alpha = 0, 1, \ldots, R\right\}$ be a family of $R + 1$ π-vertical vector fields along $\pi_{1,0}$ and $g(t)$ an arbitrary function in $\mathbb{R}$. Then, if X_g is the π-vertical vector field along $\pi_{1,0}$,

$$X_g = \epsilon \sum_{\alpha=0}^{R} \frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha} X_\alpha \tag{4.6}$$

its first prolongation X^1_g is the infinitesimal generator of the gauge transformation (4.5). Such X_g is said to be a gauge symmetry of the Lagrangian L if there exists a function $F_g \in C^\infty(\mathbb{R} \times \mathrm{T}Q)$ such that

$$d_{X^1_g} L = X^1_g L = \frac{\mathrm{d}F_g}{\mathrm{d}t}. \tag{4.7}$$

In such case $\langle \delta L, X_g\rangle + \mathrm{d}G_g/\mathrm{d}t = 0$, with $G_g = F_g - \langle \Theta_L, X_g\rangle$, and conversely, if there exists a function G_g such that $\langle \delta L, X_g\rangle + \mathrm{d}G_g/\mathrm{d}t = 0$, then $\mathrm{d}_{X^1_g} L = \mathrm{d}F_g/\mathrm{d}t$ with $F_g = G_g + \langle \Theta_L, X_g\rangle$. (The contractions $\langle \delta L, X_g\rangle$ and $\langle \Theta_L, X_g\rangle$ make sense because of the $\pi_{2,0}$- and $\pi_{1,0}$-semibasic character of δL and Θ_L, respectively.)

This entire mathematical apparatus can be used to give a geometric version of Noether's second theorem as it can be seen in [13, 28]. The theorem essentially establishes that a gauge-invariant Lagrangian is necessarily singular and it satisfies the so-called Noether identities

$$\sum_\alpha (-1)^\alpha \frac{\mathrm{d}^\alpha \langle \delta L, X_\alpha\rangle}{\mathrm{d}t^\alpha} = 0$$

i.e.

$$\sum_{i,\alpha} (-1)^\alpha \frac{\mathrm{d}^\alpha \left(L_i X^i_\alpha\right)}{\mathrm{d}t^\alpha} = 0$$

with L_i being the variational derivatives of the Lagrangian L. However, a singular Lagrangian needs not be gauge invariant and a method for the determination of the gauge symmetry underlying a given singular Lagrangian has been developed in [13] (see also [28]). The method is based on the necessary conditions which are derived from the second Noether theorem and it also tells us whether, or not, a given Lagrangian is gauge invariant. Let X_g

given by (4.6) be the wanted gauge symmetry of L and choose a vector field $\tilde{X}_\alpha \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}Q)$ such that $\mathrm{T}\pi_{1,0} \circ \tilde{X}_\alpha = X_\alpha$. Assume that

$$G_g = \sum_{\alpha=0}^{R-1} G_\alpha \frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha}.$$

It follows from the symmetry condition (4.7) that the vector field X_R belongs to the distribution $M_L - \operatorname{Ker} S$ and the functions G_α satisfy the relations

$$G_R = 0 \qquad \dot{G}_\alpha + G_{\alpha-1} + \langle \delta L, X_\alpha \rangle = 0 \quad (\alpha = 1, \ldots, R) \qquad \dot{G}_0 + \langle \delta L, X_0 \rangle = 0$$

from which it follows that all the functions G_α must be $\mathcal{F}L$-projectable and satisfy the recursive relations

$$G_{\alpha-1} = -\mathrm{d}\Theta_L(D, \tilde{X}_\alpha) - D(G_\alpha)$$

with D being any SODE; in particular, $G_{R-1} = -\mathrm{d}\Theta_L(D, \tilde{X}_R)$ is a primary constraint function (see (4.3)). The algorithm for the determination of the gauge symmetries proceeds by determining in an iterative and orderly way the functions G_α and the vector fields X_α along the following steps:

(1) Choose $G_R = 0$ and select $\tilde{X}_R \in (M_L - \operatorname{Ker} S) \cap \operatorname{Ker} \mathrm{T}\pi_1$ in such a way that $G_{R-1} = -\mathrm{d}\Theta_L(D, \tilde{X}_R)$, with D being any SODE, is a $\mathcal{F}L$-projectable primary constraint function.
(2) Then, let us determine a π_1-vertical vector field $\tilde{X}_{R-1}$ in such a way that the 1-form $\lambda_{R-1} = i(\tilde{X}_{R-1})\,\mathrm{d}\Theta_L - \mathrm{d}G_{R-1}$ be $\pi_{1,0}$-semibasic; G_{R-2} is defined by $G_{R-2} = \lambda_{R-1}(D)$.
(3) When the successive functions G_α are $\mathcal{F}L$-projectable the process may be iterated and when we find $G_{R-N} = G_R = 0$ the algorithm enters into a cycle and the solutions appear cyclically repeated. We can take $R = N - 1$ to be the higher order for the derivatives of $g(t)$ and the algorithm ends up.

In the case when in some step there is no solution, we stop the process and return to make (if possible) a new choice for the solution in a previous step. If there is no solution in any case, we have to conclude that the Lagrangian is not gauge invariant.

The following sections are devoted to show the application of these constructions to the case of a Lagrangian which is linear, or more accurately affine, in the velocities v^i.

5. Geometric theory of time-dependent Lagrangians which are affine in velocities

From the geometric viewpoint, a time-dependent Lagrangian L which is affine in the velocities arises from a 1-form $\lambda \in \bigwedge^1(\mathbb{R} \times Q)$ in the following way: $L = i_{\mathrm{T}^0}\lambda \in C^\infty(\mathbb{R} \times \mathrm{T}Q)$. In fact, if $\lambda = m_i(t,q)\,\mathrm{d}q^i - V(t,q)\,\mathrm{d}t$, then the Lagrangian, to be denoted by $\hat{\lambda}$, is

$$\hat{\lambda} = m_i(t,q)v^i - V(t,q). \tag{5.1}$$

Obviously, the time-independent case we have dealt with in section 2 is simply a special case of this: given $\mu \in \bigwedge^1(Q)$ and $V \in C^\infty(Q)$ then we consider the 1-form $\lambda = \rho^*\mu - \rho^* V dt \in \bigwedge^1(\mathbb{R} \times Q)$ which yields (ρ_2-pull-back of) the time-independent Lagrangian (2.1).

Coming back to the general case, the basic geometric features for this Lagrangian are as follows:

(1) The Poincaré–Cartan 1-form is given by $\Theta_{\hat{\lambda}} = \pi_{1,0}^*\lambda$.
(2) $\mathcal{F}\hat{\lambda}$-projectability means $\pi_{1,0}$-projectability, because

$$\mathcal{F}\hat{\lambda} = \mu \circ \lambda \circ \pi_{1,0}$$

where $\mu = \pi_{\mathbb{R}} \times \mathrm{id}_{\mathrm{T}^*Q}$ is the natural projection of $\mathrm{T}^*(\mathbb{R} \times Q)$ onto $\mathbb{R} \times \mathrm{T}^*Q$.

(3) $\mathcal{V}_{1,0}(\operatorname{Ker} d\Theta_{\hat{\lambda}}) = \operatorname{Ker} T\pi_{1,0}$, so that $M_{\hat{\lambda}} = T(\mathbb{R} \times TQ)$.

Taking into account all these facts, we find that the primary constraint functions are given by

$$\Phi_Z = d\Theta_{\hat{\lambda}}(D, Z) \qquad Z \in T(\mathbb{R} \times TQ) \qquad D \text{ any SODE} \tag{5.2}$$

i.e. $\Phi_Z = i(Z)(i_{T^0}\, d\lambda)$. The primary constraint manifold is described in the following terms:

$$\mathcal{M} = \{z \in \mathbb{R} \times TQ \mid \Phi_Z(z) = 0\} \equiv \{z \in \mathbb{R} \times TQ \mid i_{T^0}\, d\lambda(z) = 0\}. \tag{5.3}$$

However, if we recall that there exists a local basis of $\mathfrak{X}(\mathbb{R} \times TQ)$ which is made up of a SODE and vector fields Y^1 and $Y^V = S(Y^1)$, with $Y \in \operatorname{Ker} T\pi$ [28], we see that the only effective constraint functions are those given by

$$\Phi_Y = d\Theta_{\hat{\lambda}}(Y^1, D) \qquad Y \in \operatorname{Ker} T\pi. \tag{5.4}$$

Then the functions Φ_Y are in a one-to-one correspondence with the elements of $\operatorname{Ker} T\pi$. They also verify the property

$$\Phi_{Y_1 + fY_2} = \Phi_{Y_1} + \left(\pi^*_{1,0} f\right)\Phi_{Y_2} \qquad f \in C^\infty(\mathbb{R} \times Q). \tag{5.5}$$

We will say that Φ_Y and $\Phi_{\bar{Y}}$ are 'linearly dependent' if $\Phi_{\bar{Y}} = \Phi_{fY}$ for some $f \in C^\infty(\mathbb{R} \times Q)$ everywhere non-null. This property trivially takes place when $\bar{Y} = fY$ but in the case when $\bar{Y}$ and Y are not dependent it is a property related with the existence of a gauge symmetry of the Lagrangian as we will see later.

In local coordinates (t, q^i) for $\mathbb{R} \times Q$ and (t, q^i, v^i) for $\mathbb{R} \times TQ$, respectively, the vector field Y is written as $Y = Y^i(t, q)\partial/\partial q^i$ and $d\lambda = \frac{1}{2} A_{ij}\, dq^i \wedge dq^j - \omega_i\, dq^i \wedge dt$, where

$$A_{ij} = \frac{\partial m_j}{\partial q^i} - \frac{\partial m_i}{\partial q^j} \quad \text{and} \quad \omega_i = \frac{\partial V}{\partial q^i} + \frac{\partial m_i}{\partial t}. \tag{5.6}$$

Consequently, the primary constraint functions are

$$\Phi_Y = (A_{ij} v^j - \omega_i) Y^i \tag{5.7}$$

i.e. they are affine in the velocities. A basis for such constraint functions is made up of the following functions:

$$\Phi_i = \Phi_{\partial/\partial q^i} = A_{ij} v^j - \omega_i = 0 \qquad i = 1, \ldots, n. \tag{5.8}$$

Note that these equations are the Euler–Lagrange equations obtained from the Lagrangian (5.1) and all of them appear in this formalism as constraint equations.

We can deduce from (5.4) that a primary constraint function Φ_Y is $\pi_{1,0}$-projectable (i.e. holonomic) if and only if the 1-form $i(Y)\, d\lambda$ is π-semibasic. In an equivalent way, Φ_Y is $\pi_{1,0}$-projectable if and only if

$$Y \in \Omega_\lambda = [\operatorname{Ker} T\pi]^{\perp\, d\lambda} \cap \operatorname{Ker} T\pi$$

(the superscript $\perp$ $d\lambda$ means $d\lambda$-orthogonal complement). In local coordinates as above, Ω_λ is spanned by vector fields $Y = Y^i(t, q)\partial/\partial q^i$ such that $A_{ij} Y^i = 0$ and then the corresponding constraint function $\Phi_Y = \omega_i Y^i = 0$ is projectable.

Obviously, the maximum number of linearly independent constraint functions equals the dimension n of Q and the maximum number of the holonomic ones is $p = n - \operatorname{rank} A$, where A is the skew-symmetric matrix of elements A_{ij} (5.6).

As far as the dynamics is concerned, any SODE Γ is a solution on $\mathcal{M}$ of the dynamical equation because of (5.2). The consistency conditions (4.4), $\Gamma\Phi_Y = 0$, will generate additional (secondary) constraints and/or fix (maybe partially) the dynamics. More accurately, the consistency conditions for non-projectable primary constraint functions will determine some components of the SODE, while the projectable ones will give rise to secondary constraints that are affine in the velocities, which fix the dynamics. The uniqueness is obtained when all the constraints so obtained are independent and in the maximum number. The two fundamental cases are as follows.

(I) The first fundamental case arises when $\Omega_\lambda = 0$. Then the matrix A is regular and therefore there are no primary holonomic constraints. The n independent constraints (5.8) are nonholonomic and lead to n equations determining, in a unique way, the dynamics on $\mathcal{M}$, i.e.

$$v^j = (A^{-1})^{jk}\omega_k. \tag{5.9}$$

Of course, this is only possible when the dimension of Q is even, $n = 2m$.

Let us analyse this 'regular' case in geometrical terms. The condition $\Omega_\lambda = 0$ means that the distribution $\operatorname{Ker} \mathrm{d}\lambda$ is one dimensional, so $\mathrm{d}\lambda$ is a contact form on $\mathbb{R} \times Q$ generating an exact contact structure on $\mathbb{R} \times Q$. In fact, every nonzero vector field $Z \in \operatorname{Ker} \mathrm{d}\lambda$ satisfies the condition $i(Z)\,\mathrm{d}t \neq 0$, as it follows from the trivial fact that Z also belongs to $[\operatorname{Ker} \mathrm{T}\pi]^{\perp\,\mathrm{d}\lambda}$, and if the condition $i(Z)\,\mathrm{d}t = 0$ is fulfilled Z should also be π-vertical and, consequently, $Z = 0$. Moreover for every pair $Z, Z' \in \operatorname{Ker} \mathrm{d}\lambda$ the vector field $Z'' = [i(Z)\,\mathrm{d}t]Z' - [i(Z')\,\mathrm{d}t]Z$ lies in Ω_λ, $Z'' \in \Omega_\lambda$, i.e. $Z' = [i(Z')\,\mathrm{d}t/i(Z)\,\mathrm{d}t]Z$.

On the other hand, no new constraints arise from the consistency conditions (4.4) and the n-independent nonholonomic constraints will fix the dynamics on $\mathcal{M}$ in a unique way. Let $\eta \in \operatorname{Ker} \mathrm{d}\lambda$ be a vector field such that $i(\eta)\,\mathrm{d}t = 1$. It generates locally $\operatorname{Ker} \mathrm{d}\lambda$ and in coordinates turns out to be

$$\eta = \frac{\partial}{\partial t} + (A^{-1})^{ij}\omega_j \frac{\partial}{\partial q^i}. \tag{5.10}$$

The description (5.3) of $\mathcal{M}$ means that $\mathbf{T}^0(z) \in \operatorname{Ker} \mathrm{d}\lambda(\pi_{1,0}(z))$, $z\in\mathcal{M}$, and, consequently, we can assert that the ICT η^1 is, at least on $\mathcal{M}$, a SODE field. It is such that $i(\eta^1)\,\mathrm{d}\Theta_{\hat\lambda}|_{\mathcal{M}} = 0$, thus the dynamical equation (4.2) is satisfied by η^1 on $\mathcal{M}$ and the SODE Γ compatible with the constraints is that generated by η^1.

In summary, when $\Omega_\lambda = 0$, the Lagrangian system on $\mathbb{R} \times \mathrm{T}Q$ reduces to the (in general, time-dependent) Hamiltonian system $(\mathbb{R} \times Q,\ \mathrm{d}\lambda,\ H)$. The manifold $\mathbb{R} \times Q$ is the extended phase space and the Hamiltonian function H is essentially the energy $E_{\hat\lambda}$, which is a holonomic function: $E_{\hat\lambda} = \pi_{1,0}^* H$; in local coordinates $H = V$.

Both dynamical systems are equivalent in the sense that the integral curves of the dynamics in $\mathbb{R} \times \mathrm{T}Q$ are the first prolongation to $\mathbb{R} \times \mathrm{T}Q$ of the integral curves of the Hamiltonian dynamics in $\mathbb{R} \times Q$.

Using the 2-form $\mathrm{d}\lambda$ we can define a Poisson bracket in $\mathbb{R} \times Q$ according to the following construction. Given a 1-form $\alpha \in \bigwedge^1(\mathbb{R} \times Q)$ there exists a vector field $X_\alpha \in \mathfrak{X}(\mathbb{R} \times Q)$ such that $i(X_\alpha)\,\mathrm{d}\lambda = \alpha$ iff $i(Z)\alpha = 0, \forall Z \in \operatorname{Ker} \mathrm{d}\lambda$; obviously such a X_α is not unique, the indeterminacy being $\operatorname{Ker} \mathrm{d}\lambda$ itself. Although α does not satisfy the condition above one can take the '$\mathrm{d}\lambda$-semibasic' part, given by

$$\alpha^\lambda = \alpha - \frac{i(Z)\alpha}{i(Z)\,\mathrm{d}t}\,\mathrm{d}t \qquad Z \in \operatorname{Ker} \mathrm{d}\lambda \tag{5.11}$$

which, in fact, depends only on $\operatorname{Ker} \mathrm{d}\lambda$ and not on a particular $Z \in \operatorname{Ker} \mathrm{d}\lambda$, and it trivially annihilates $\operatorname{Ker} \mathrm{d}\lambda$, i.e. there exist vector fields X_α such that $i(X_\alpha)\,\mathrm{d}\lambda = \alpha^\lambda$.

In local coordinates, if $\alpha = a_0\,\mathrm{d}t + a_i\,\mathrm{d}q^i$, then

$$a^\lambda = a_i(\mathrm{d}q^i - (A^{-1})^{ij}\omega_j\,\mathrm{d}t)$$

and

$$X_\alpha = X^0\eta + (A^{-1})^{jk}\alpha_j \frac{\partial}{\partial q^k}$$

where X^0 is an arbitrary function and η is given by (5.10); in particular, for $\alpha = \mathrm{d}f$,

$$\mathrm{d}^\lambda f = (\mathrm{d}f)^\lambda = \frac{\partial f}{\partial q^i}(\mathrm{d}q^i - (A^{-1})^{ij}\omega_j\,\mathrm{d}t)$$

and

$$X_f = X_{\mathrm{d}f} = X^0\eta + (A^{-1})^{jk}\frac{\partial f}{\partial q^j}\frac{\partial}{\partial q^k}.$$

The Poisson bracket $\{f, g\}$ of the functions f and g is then defined by the rule

$$\{f, g\} = \mathrm{d}\lambda(X_g, X_f) = \mathrm{d}^\lambda g(X_f). \tag{5.12}$$

In local coordinates,

$$\{f, g\} = (A^{-1})^{jk}\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial q^k}$$

the fundamental Poisson brackets being $\{q^j, q^k\} = (A^{-1})^{jk}$ (see [42]). So the equation of motion derived from the constraint equations (5.8) can be written in the form $\dot{q}^j = \{q^j, q^k\}\omega_k$.

In the autonomous case $(Q, -\mathrm{d}\mu)$ is a symplectic manifold and we obtain the Hamiltonian system $(Q, -\mathrm{d}\mu, V)$ analysed in detail in section 2. In particular, the equations of motion read

$$\dot{q}^i = \{q^i, q^j\}\frac{\partial V}{\partial q^j}$$

(see (2.9)).

(II) Second case: $\Omega_\lambda \neq 0$. Then A is singular and there will be $r = \operatorname{rank} A$ independent non-projectable constraint functions, and then the number of effective projectable constraint functions is not greater than $p = n - r$. These functions generate secondary constraint functions that together with the non-projectable functions will determine, at least partially, the dynamics. The uniqueness of solution for the dynamics depends on whether or not the primary and secondary constraints are independent. The analysis of gauge invariance in the following section will clarify these points.

6. Gauge invariance of affine in velocities Lagrangian systems

In this section, we will show how to make use of the algorithm of gauge symmetry explained in section 4. It starts by taking $G_R = 0$ and choosing a vector field $\tilde{X}_R \in (M_{\hat{\lambda}} - \operatorname{Ker} S) \cap \operatorname{Ker} \mathrm{T}\pi_1$ in such a way that the function G_{R-1} given by $G_{R-1} = \mathrm{d}\Theta_{\hat{\lambda}}(\tilde{X}_R, D)$ be a $\pi_{1,0}$-projectable primary constraint function. Then, we can choose $\tilde{X}_R = Y^1$ (that is, $X_R = \mathrm{T}\pi_{1,0} \circ Y^1 = Y \circ \pi_{1,0}$, where $Y \in \Omega_\lambda$), so that G_{R-1} turns out to be $G_{R-1} = \Phi_Y$ that is a $\pi_{1,0}$-projectable function. Moreover, we can see that the 1-form $\lambda_{R-1} = i(\tilde{X}_{R-1})\,\mathrm{d}\Theta_{\hat{\lambda}} - \mathrm{d}G_{R-1}$ is $\pi_{1,0}$-semibasic, irrespective of the choice of $\tilde{X}_{R-1}$ is. We can write $\tilde{X}_{R-1}$ as a sum

$$\tilde{X}_{R-1} = Y_1^1 + Y_2^V \qquad Y_1, Y_2 \in \operatorname{Ker} \mathrm{T}\pi; \tag{6.1}$$

and, consequently,

$$G_{R-2} = \mathrm{d}\Theta_{\hat{\lambda}}\big(Y_1^1, D\big) - D(\Phi_Y) = \Phi_{Y_1} - \chi_Y \tag{6.2}$$

is the difference between a primary constraint function and a secondary one. The algorithm only works if it is possible to choose G_{R-2} as being a $\pi_{1,0}$-projectable function. The iteration of this procedure will give rise to a sequence of functions G_{R-k} analogous to G_{R-2}, and therefore a gauge symmetry will be obtained in this way if we arrive at a secondary constraint function χ_Y that was a primary one, making possible to have $G_{R-2} = 0$.

Thus the starting point to have a gauge symmetry is to know the set of holonomic (primary) constraint functions. If two (or more) of them are linearly dependent it will be possible to choose X_R in such a way that $G_{R-1} = 0$. In the opposite case, we need to know whether or not the primary constraint function gives rise to a secondary constraint function which is a primary one.

More specifically, there are the following four possibilities and correspondingly the four types of affine in velocities Lagrangians:

(1) $\Omega_\lambda = 0$, i.e. A is regular. In this case there are no holonomic constraints and there is no gauge symmetry at all. The dynamics is uniquely determined by the nonholonomic constraints (see section 5).

(II.1) $\Omega_\lambda \neq 0$, i.e. A is singular, and there are two (non-trivially) linearly dependent holonomic primary constraint functions $\Phi_{\bar{Y}}$ and Φ_Y, i.e. $\Phi_{\bar{Y}} = \Phi_{fY} = (\pi_{1,0}^* f)\Phi_Y$. In this case, $\bar{Y} - fY$ is non-null and choosing $\tilde{X}_R$ as $\tilde{X}_R = (\bar{Y} - fY)^1$, we get $G_{R-1} = 0$ (i.e., the higher order for the derivatives of the arbitrary function $g(t)$ is $R = 0$) and come to the gauge symmetry

$$X_g = g(\bar{Y} - fY) \circ \pi_{1,0}. \tag{6.3}$$

(II.2) $\Omega_\lambda \neq 0$ and there is a holonomic primary constraint function Φ_Y, $0 \neq Y \in \Omega_\lambda$, giving rise to a secondary constraint function $\chi_Y = D(\Phi_Y)$ which is a primary one, namely, there exists a vector field $\bar{Y} \in \operatorname{Ker} \mathrm{T}\pi$ such that $\chi_Y = \Phi_{\bar{Y}} = \mathrm{d}\Theta_{\hat{\lambda}}(\bar{Y}^1, D)$. In this case, choosing $Y_1 = \bar{Y}$ we find that $G_{R-2} = \mathrm{d}\Theta_{\hat{\lambda}}(\bar{Y}^1, D) - D(\Phi_Y) = 0$. Therefore the algorithm tells us that $R = 1$ and the gauge symmetry is

$$X_g = g(\bar{Y} \circ \pi_{1,0}) + \dot{g}(Y \circ \pi_{1,0}). \tag{6.4}$$

(II.3) $\Omega_\lambda \neq 0$ and none of the secondary constraints is primary. G_{R-2} is not holonomic and the algorithm cannot go on, i.e. there is no gauge symmetry. The dynamics is uniquely determined by the full set of constraints (both primary and secondary).

In summary, we will have gauge symmetry when there exist holonomic constraint functions generating secondary constraint functions that generate a free set with the non-projectable constraint functions.

7. Examples

Finally, several examples will be used to illustrate the Lagrangian analysis made in sections 5 and 6. As a matter of notation, we will use subindices instead of superindices in the coordinates q and the velocities v.

Example 1. The well-known two-dimensional Lotka–Volterra system can be derived from the following Lagrangian which is affine in the velocities [43]:

$$L = \frac{\ln y}{2x} v_x - \frac{\ln x}{2y} v_y - (a \ln y + b \ln x - x - y)$$

where a and b are positive constants. Considered as a Lagrangian function on $\mathbb{R} \times \mathrm{T}\,\mathbb{R}^2_+$ it derives from the 1-form

$$\lambda = \frac{\ln y}{2x}\,\mathrm{d}x - \frac{\ln x}{2y}\,\mathrm{d}y - (a \ln y + b \ln x - x - y)\,\mathrm{d}t \in \bigwedge^1 \left(\mathbb{R} \times \mathbb{R}^2_+\right).$$

Therefore,

$$A = \begin{pmatrix} 0 & -\frac{1}{xy} \\ \frac{1}{xy} & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \frac{b}{x} - 1 \\ \frac{a}{y} - 1 \end{pmatrix}.$$

The matrix A is regular and then $\Omega_\lambda = 0$. All of the primary constraint functions (5.8) are nonholonomic,

$$\Phi_x = \Phi_{\partial/\partial x} = -\frac{v_y}{xy} - \frac{b}{x} + 1 \qquad \Phi_y = \Phi_{\partial/\partial y} = \frac{v_x}{xy} - \frac{a}{y} + 1$$

and, consequently, the reduced system (5.10) on $\mathbb{R} \times \mathbb{R}^2_+$ is

$$\eta = \frac{\partial}{\partial t} + x(a-y)\frac{\partial}{\partial x} - y(b-x)\frac{\partial}{\partial y}.$$

The system of differential equations for its integral curves constitutes the two-dimensional Lotka–Volterra system

$$\dot{x} = x(a-y) \qquad \dot{y} = -y(b-x).$$

This is a Hamiltonian system with a symplectic structure

$$\sigma = \frac{1}{xy}\,\mathrm{d}y \wedge \mathrm{d}x$$

i.e. with defining Poisson bracket $\{x, y\} = xy$ and Hamiltonian function

$$H = a\ln y + b\ln x - x - y.$$

This Lagrangian is of type I.

Example 2. Another regular case is that provided by the Lagrangian studied in [44],

$$L = \tfrac{1}{2}[(q_2+q_3)v_1 - q_1v_2 + (q_4-q_1)v_3 - q_3v_4] - \tfrac{1}{2}\big(2q_2q_3 + q_3^2 + q_4^2\big)$$

coming from the 1-form $\lambda \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^4)$ given by

$$\lambda = \tfrac{1}{2}\big[(q_2+q_3)\,\mathrm{d}q_1 - q_1\,\mathrm{d}q_2 + (q_4-q_1)\,\mathrm{d}q_3 - q_3\,\mathrm{d}q_4\big] - \tfrac{1}{2}\big(2q_2q_3 + q_3^2 + q_4^2\big)\,\mathrm{d}t$$

from which we obtain

$$A = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 \\ q_3 \\ q_2+q_3 \\ q_4 \end{pmatrix}.$$

The matrix A has rank 4 and then, as $\Omega_\lambda = 0$, there is no holonomic constraint. The generating set of primary constraints (5.8) is

$$\Phi_1 = -v_2 - v_3 = 0 \quad \Phi_2 = v_1 - q_3 = 0 \quad \Phi_3 = v_1 - v_4 - q_2 - q_3 = 0 \quad \Phi_4 = v_3 - q_4 = 0$$

and the secondary ones, $\Gamma(\Phi_i) = 0$, with $i = 1, \ldots, 4$, determine a unique dynamics which is the restriction of a SODE on the constraint manifold $\mathcal{M}$, namely

$$\Gamma' = \Gamma_{|\mathcal{M}} = \frac{\partial}{\partial t} + q_3\frac{\partial}{\partial q_1} - q_4\frac{\partial}{\partial q_2} + q_4\frac{\partial}{\partial q_3} - q_2\frac{\partial}{\partial q_4} + v_3\frac{\partial}{\partial v_1} - v_4\frac{\partial}{\partial v_2} + v_4\frac{\partial}{\partial v_3} - v_2\frac{\partial}{\partial v_4}.$$

The reduced system on $\mathbb{R} \times \mathbb{R}^4$ is

$$\eta = \frac{\partial}{\partial t} + q_3\frac{\partial}{\partial q_1} - q_4\frac{\partial}{\partial q_2} + q_4\frac{\partial}{\partial q_3} - q_2\frac{\partial}{\partial q_4} \in \operatorname{Ker}\mathrm{d}\lambda.$$

Note that the restriction of Γ' onto $\mathcal{M}$ coincides with that of η^1.

Example 3. Let us now consider the Lagrangian of the type II.1

$$L = q_1v_2 + q_2v_3 + q_2v_4 - q_2(q_4 - q_3)$$

generated by the 1-form

$$\lambda = q_1\,\mathrm{d}q_2 + q_2\,\mathrm{d}q_3 + q_2\,\mathrm{d}q_4 - q_2(q_4 - q_3)\,\mathrm{d}t \in \bigwedge^1(\mathbb{R}\times\mathbb{R}^4)$$

from which we obtain

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 \\ q_4 - q_3 \\ -q_2 \\ q_2 \end{pmatrix}.$$

The primary constraint functions (5.8) are

$$\Phi_1 = v_2 \qquad \Phi_2 = -v_1 + v_3 + v_4 - q_4 + q_3 \qquad \Phi_3 = -v_2 + q_2 \qquad \Phi_4 = -v_2 - q_2.$$

The distribution Ω_λ is spanned by the two vector fields $\partial/\partial q_1 + \partial/\partial q_3$ and $\partial/\partial q_1 + \partial/\partial q_4$, and yields two holonomic primary constraint functions

$$\Phi_1 + \Phi_3 = q_2 \qquad \Phi_1 + \Phi_4 = -q_2$$

that are linearly dependent. The Lagrangian is gauge invariant, and the symmetry vector (6.3) and the associated function F_g are $(R = 0)$

$$X_g = g(t)\left(2\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_3} + \frac{\partial}{\partial q_4}\right) \quad \text{and} \quad F_g = 2g(t)q_2.$$

The dynamics Γ is determined by the set of (primary and secondary) constraints $\Phi_i, \chi_i, i = 1, \ldots, 4$,

$$\Gamma = \frac{\partial}{\partial t} + (v_3 + v_4 - q_4 + q_3)\frac{\partial}{\partial q_1} + v_3\frac{\partial}{\partial q_3} + v_4\frac{\partial}{\partial q_4} + (C_3 + C_4 - v_4 + v_3)\frac{\partial}{\partial v_1} + C_3\frac{\partial}{\partial v_3} + C_4\frac{\partial}{\partial v_4}$$

where C_3 and C_4 are arbitrary functions.

Example 4. The Lagrangian defined by the 1-form

$$\lambda = (q_2 - q_3)\,\mathrm{d}q_1 - q_2\,\mathrm{d}q_3 - (q_2 - q_1)q_3\,\mathrm{d}t \in \bigwedge^1(\mathbb{R}\times\mathbb{R}^3)$$

is of type II.2. The matrices A and ω are given, respectively, by

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} -q_3 \\ q_3 \\ q_2 - q_1 \end{pmatrix}.$$

The set of primary constraints (5.8) is generated by

$$\Phi_1 = -v_2 + v_3 + q_3 \qquad \Phi_2 = v_1 - v_3 - q_3 \qquad \Phi_3 = -v_1 + v_2 - q_2 + q_1.$$

The rank of A is 2 and therefore the distribution Ω_λ is one dimensional. It is generated by the vector field $\partial/\partial q_1 + \partial/\partial q_2 + \partial/\partial q_3$ and there is one primary holonomic constraint function, namely, $\Phi_1 + \Phi_2 + \Phi_3 = q_1 - q_2$, whose corresponding secondary constraint function, $\chi = v_1 - v_2$, is likewise a primary constraint: $\chi = \Phi_{\partial/\partial q_1 + \partial/\partial q_2} = \Phi_1 + \Phi_2$. Consequently, this Lagrangian is gauge invariant: starting from $X_R = \partial/\partial q_1 + \partial/\partial q_2 + \partial/\partial q_3$ and choosing $X_{R-1} = \partial/\partial q_1 + \partial/\partial q_2$ we obtain the gauge symmetry (6.4)

$$X_g = g(t)\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right) + \dot{g}(t)\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}\right)$$

and the corresponding function $F_g = g(t)(q_1 - q_3) - \dot{g}(t)q_3$. The dynamics on $\mathcal{M}$ is given by

$$\Gamma = \frac{\partial}{\partial t} + (v_3 + q_3)\frac{\partial}{\partial q_1} + (v_3 + q_3)\frac{\partial}{\partial q_2} + v_3\frac{\partial}{\partial q_3} + (C + v_3)\frac{\partial}{\partial v_1} + (C + v_3)\frac{\partial}{\partial v_2} + C\frac{\partial}{\partial v_3}$$

with C an arbitrary function.

Example 5. The 1-form

$$\lambda = tq_2\,\mathrm{d}q_1 - q_1\,\mathrm{d}q_2 - [q_1 - (t+1)q_2]q_3\,\mathrm{d}t \in \bigwedge{}^1(\mathbb{R}\times\mathbb{R}^3)$$

gives the time-dependent Lagrangian of type II.2,

$$\hat{\lambda} = tq_2v_1 - q_1v_2 - [q_1 - (t+1)q_2]q_3.$$

The matrices A and ω are given, respectively, by

$$A = \begin{pmatrix} 0 & -(t+1) & 0 \\ t+1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} q_2 + q_3 \\ -(t+1)q_3 \\ q_1 - (t+1)q_2 \end{pmatrix}.$$

Two constraint functions of those determined by (5.8) are nonholonomic,

$$\Phi_1 = -(t+1)v_2 - q_2 - q_3 \qquad \Phi_2 = (t+1)v_1 + (t+1)q_3$$

while Φ_3 is holonomic,

$$\Phi_3 = -q_1 + (t+1)q_2.$$

It gives rise to a secondary constraint, $\chi_3 = -v_1 + (t+1)v_2 + q_2$, which is a primary one, $\chi_3 = -\Phi_1 - \Phi_2/(t+1)$. The constraint function χ_3 corresponds to the vector field $-\partial/\partial q_1 - (t+1)\partial/\partial q_2$ and, consequently, $\hat{\lambda}$ is gauge invariant. The symmetry vector X_g (6.4) and the function F_g are ($R = 1$)

$$X_g = g(t)\left(-\frac{\partial}{\partial q_1} - \frac{1}{t+1}\frac{\partial}{\partial q_2}\right) + \dot{g}(t)\frac{\partial}{\partial q_3} \qquad F_g = g(t)\left(q_2 - \frac{t}{t+1}q_1\right).$$

The local expression for the dynamical vector field is

$$\Gamma = \frac{\partial}{\partial t} - q_3\frac{\partial}{\partial q_1} - \frac{q_2 + q_3}{t+1}\frac{\partial}{\partial q_2} + v_3\frac{\partial}{\partial q_3} - v_3\frac{\partial}{\partial v_1} - \frac{2v_2 + v_3}{t+1}\frac{\partial}{\partial v_2} + C\frac{\partial}{\partial v_3}$$

with C being an arbitrary function.

Example 6. A Lagrangian of the type II.3 is that provided by a slight modification of the 1-form λ in example 4:

$$\lambda = (q_2 - q_3)\,\mathrm{d}q_1 + q_2\,\mathrm{d}q_3 + (q_1 - q_2)q_3\,\mathrm{d}t \in \bigwedge^1(\mathbb{R}\times\mathbb{R}^3).$$

In this case,

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} -q_3 \\ q_3 \\ q_2 - q_1 \end{pmatrix}.$$

The primary constraint functions given by (5.8) are

$$\Phi_1 = -v_2 + v_3 + q_3 \qquad \Phi_2 = v_1 + v_3 - q_3 \qquad \Phi_3 = -v_1 - v_2 + q_1 - q_2.$$

The distribution Ω_λ is spanned by the vector field $\partial/\partial q_1 - \partial/\partial q_2 - \partial/\partial q_3$. There is a holonomic constraint function, $\Phi_1 - \Phi_2 - \Phi_3 = 2q_3 - q_1 + q_2$, giving rise to a secondary one, $\chi = 2v_3 - v_1 + v_2$, which is not primary, that is, there is no π-vertical vector field Y such that $\chi = \Phi_Y$. Consequently, there is no gauge symmetry at all, and the dynamics Γ on the constraint manifold $\mathcal{M}$ is unique:

$$\Gamma = \frac{\partial}{\partial t} + q_3\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right) + v_3\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right).$$

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